

Hydrodynamic flow of ions and atoms in partially ionized plasmasR. A. Nemirovsky,^{1,*} D. R. Fredkin,² and A. Ron¹¹*Department of Physics, Technion-Israel Institute of Technology, Haifa 32000, Israel*²*Department of Physics, University of California, San Diego, La Jolla, California 92093*

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We have derived the hydrodynamic equations of motion for a partially ionized plasma, when the ionized component and the neutral components have different flow velocities and kinetic temperatures. Starting from the kinetic equations for a gas of ions and a gas of atoms we have considered various processes of encounters between the two species: self-collisions, interspecies collisions, ionization, recombination, and charge exchange. Our results were obtained by developing a general approach for the hydrodynamics of a gas in a binary mixture, in particular when the components drift with respect to each other. This was applied to a partially ionized plasma, when the neutral-species gas and the charged-species gas have separate velocities. We have further suggested a generalized version of the relaxation time approximation and obtained the contributions of the interspecies encounters to the transport equations.

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I. INTRODUCTION

Neutral atoms play an important role in the dynamics of partially ionized gases when electromagnetic forces are present. Since they are unaffected by these forces, the atoms exert an effective drag on the ions and electrons, and may even flow separately. The effect of neutral atoms was shown to be significant in various laboratory plasmas, such as *Z-pinch* plasma [1] and *tokamak* plasma [2,3]. In nature, this effect was shown to be dominant during processes of star formation in stellar gas clouds [4–7]. The hydrodynamics equations of motion in the case of separate flow of neutral atoms is usually obtained by considering the plasma as a mixture of two interacting gaseous fluids, one consisting of ions and electrons (*charged-species fluid*), and the other consisting of neutral atoms (*neutral-species fluid*) [1,3,8]. The interaction between the fluids is introducing additional terms in the flow equations for each fluid. The present paper is concerned with the derivation of these contributions to the separate fluids equations of motion.

The hydrodynamics of a gas, or gas mixture, is normally obtained by means of some variation of the Chapman-Enskog or Hilbert method [9–11]. The hydrodynamic limit occurs when collisions dominate the transport equation. The essence of the procedure is that quantities conserved in collisions are identified, and a local equilibrium distribution function, for which the entropy production vanishes is identified (Boltzmann's *H* theorem). The distribution function is expanded in a series whose initial term is the local equilibrium distribution function. To first order, the collision term is linearized around local equilibrium, and the streaming terms in the transport equation contain only the local equilibrium distribution function. The condition that the resulting linear integral equation be solvable is that the moments of the transport equation corresponding to the quantities conserved in collisions vanish.

In the case of a binary mixture, without chemical reac-

tions (ionization and recombination in our case of ions and neutral atoms [3]), the quantities conserved in collisions are the densities of the two particle species, the *total* momentum, and the *total* energy. When the local equilibrium distribution is constructed, Lagrange multipliers corresponding to the total momentum density and the total energy density appear, and these are the hydrodynamic velocity and the kinetic temperature.

Traditionally, binary gas mixtures were studied assuming the mixture is having a common single temperature for both gases [9,10], or common single flow velocity like in regular *magnetohydrodynamics* (MHD). During the last two decades some attempts were made to study the separate flow of neutral species and charged species in tokamak edge plasma [2,3], assuming that charge exchange reactions, ionization, and recombination are far more frequent than elastic collisions between the ions and the neutral atoms. In the present paper we consider the effect of elastic collisions, and show that it can compare to—and even dominate over—the effect of charge exchange reactions. This occurs when the drift between the ions and the atoms is much larger than their thermal velocities. The reason that elastic collisions between atoms and ions were previously neglected was that only mild flow velocity differences were considered, as is indeed the situation in tokamak edge plasma. There are, however, interesting cases where a significant relative flow difference is established between the charged and the neutral components of the plasma. For example, we have previously studied a specific *Z-pinch* experiment, i.e., a capillary discharge *Z-pinch* for soft-x-ray amplification, and found that the relative velocity difference, which is being developed between the two species, exceeds their thermal velocities by (at least) an order-of-magnitude [1].

We start, in Sec. II, by introducing a general procedure to derive the hydrodynamics of a *general* binary mixture, when the interaction between like particles is much stronger than that between different particles, and the two components are hydrodynamically separated. In this case, we may ignore in first approximation the interspecies collisions. Under this approximation, there are four additional collisional invariants

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(the momentum and energy densities of each one of the species), and, accordingly, there are four additional Lagrange multipliers in the local equilibrium distribution: there is now a *separate* hydrodynamic velocity and kinetic temperature for each one of the species. In Sec. III we apply our general procedure to a mixture of atoms (neutral species) and ions (charged species) with significantly differing flow velocities, and find the contributions of the interactions between the two gases to the hydrodynamic equations. We then observe that interspecies elastic collisions lead to a relaxation of the separate hydrodynamic velocities and temperatures to common flow velocity and kinetic temperature. Motivated by this result we suggest, in Sec. IV, a generalized version of the *relaxation time approximation* (RTA) to the Boltzmann collision operator—in which the linearized collision term has a minimal number of nonzero eigenvalues. In the intercollision part of the new RTA model we introduce three parameters, which are chosen to be consistent with the results of Sec. III, so that they lead to the same hydrodynamic equations. We then use the suggested RTA model to refine the results of Sec. III. In Sec. V we consider the effect of three other coupling mechanisms—charge exchange, ionization, and recombination—and obtain interesting results for the contributions of charge exchange to the flow equations. Finally, in Sec. VI we summarize our results. Our main results are the expressions for the contribution of elastic collisions and charge exchange to the hydrodynamic equations of separately flowing, mixed neutral gas and charged gas in a partially ionized plasma. The effect of elastic intercollisions on the flow becomes significant when a large flow separation is developed between the components of the plasma.

II. HYDRODYNAMICS OF BINARY MIXTURE—PRINCIPLES

In this section we present the principles of the derivation of the hydrodynamic equations of a binary mixture, when the different species cannot be described in terms of a *single flow velocity*, and a *single temperature*. We start from the kinetic equations for the distribution functions $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, of the α species, where \mathbf{r} is the position, \mathbf{v} is the velocity, and t is the time. We recall that the hydrodynamic quantities, density, flow velocity, kinetic temperature, etc., are defined as velocity moments of $f_\alpha(\mathbf{r}, \mathbf{v}, t)$ —see the Appendix.

The equation of motion for $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, i.e., the *Boltzmann* kinetic equation, reads as

$$\begin{aligned} \frac{D}{Dt} f_\alpha(\mathbf{r}, \mathbf{v}, t) &= \left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} + \frac{\mathbf{F}_\alpha}{m_\alpha} \cdot \frac{\partial}{\partial \mathbf{v}} \right) f_\alpha(\mathbf{r}, \mathbf{v}, t) \\ &= \left[\frac{\partial}{\partial t} f_\alpha(\mathbf{r}, \mathbf{v}, t) \right]_{\text{coll}}, \end{aligned} \quad (1)$$

where D/Dt stands for the hydrodynamic derivative, m_α is the mass of an α -particle, and \mathbf{F}_α is the external and long range forces (such as electro-magnetic forces) acting on fluid α . The right hand side of Eq. (1) represents the influence of close range interaction (*collisions*) between the particles. While the term “collisions” may stand for various kinds of

encounters between two particles we use it here only for elastic collisions. The collisions between particles of the same species will be designated as *self-collisions* (SC), while collisions between particles of different species will be designated as *inter-collisions* (IC).

We separate the *Boltzmann collision operator* of Eq. (1) into

$$\left[\frac{\partial}{\partial t} f_\alpha(\mathbf{v}) \right]_{\text{coll}} = \sum_{\gamma=1,2} \mathcal{J}_{\alpha\gamma}[f_\alpha, f_\gamma], \quad (2)$$

where the binary operator $\mathcal{J}_{\alpha\gamma}$ is expressed by

$$\begin{aligned} \mathcal{J}_{\alpha\gamma}[f_\alpha, f_\gamma] &\equiv \int d^3\mathbf{p}_1 \int d^3\mathbf{p}' \int d^3\mathbf{p}'_1 \Gamma(\mathbf{p}, \mathbf{p}_1 \rightarrow \mathbf{p}', \mathbf{p}'_1) \delta^3(\mathbf{p} \\ &+ \mathbf{p}_1 - \mathbf{p}' - \mathbf{p}'_1) \delta \left(\frac{p^2}{2m_\alpha} + \frac{p_1^2}{2m_\gamma} - \frac{p'^2}{2m_\alpha} - \frac{p'^2_1}{2m_\gamma} \right) \\ &\times [f_1^{(0)}(\mathbf{p}') f_2^{(0)}(\mathbf{p}'_1) - f_1^{(0)}(\mathbf{p}) f_2^{(0)}(\mathbf{p}_1)]. \end{aligned} \quad (3)$$

Here $\mathbf{p} = m_\alpha \mathbf{v}$ and $\mathbf{p}_1 = m_\gamma \mathbf{v}_1$ are the momenta of the incoming α and γ particles, and $\mathbf{p}' = m_\alpha \mathbf{v}'$ and $\mathbf{p}'_1 = m_\gamma \mathbf{v}'_1$ are the momenta of the outgoing α and γ particles. When the integrations over the δ functions are performed $\mathcal{J}_{\alpha\gamma}$ can be cast into the form

$$\begin{aligned} \mathcal{J}_{\alpha\gamma}[f_\alpha, f_\gamma] &\equiv \int d^3v_1 d\Omega \sigma_{\alpha\gamma}(\Omega) |\mathbf{v} - \mathbf{v}_1| [f_\alpha(\mathbf{v}') f_\gamma(\mathbf{v}'_1) \\ &- f_\alpha(\mathbf{v}) f_\gamma(\mathbf{v}_1)]. \end{aligned} \quad (4)$$

Here $\sigma_{\alpha\gamma}(\Omega)$ is the scattering cross section between a particle of type α and a particle of type γ with scattering angle Ω . For brevity, we have omitted \mathbf{r} and t from the arguments of f in Eq. (4).

We are concerned here only with a mixture of two fluids, and write Eq. (1) for the α component as

$$\frac{D}{Dt} f_\alpha(\mathbf{r}, \mathbf{v}, t) = \mathcal{J}_{\alpha\alpha}[f_\alpha, f_\alpha] + \mathcal{J}_{\alpha\beta}[f_\alpha, f_\beta], \quad (5)$$

where $\alpha, \beta = 1$ or 2 , and $\beta \neq \alpha$. The first term on the right-hand side of Eq. (5) is the standard collision operator for a gas of single species, while the second term represents the effect of inter-collisions between the two species. Starting from the kinetic equations, the hydrodynamics equations are obtained by taking the *relevant velocity moments* of Eq. (5), i.e., the zeroth, the first, and the second. Namely, multiplying both sides of the kinetic equation by $1, m_\alpha \mathbf{v}$, and $m_\alpha v^2/2$, respectively, and integrating over the velocity $\int d^3v$ produce the equations of mass, momentum, and energy conservation. We notice that since only elastic collisions are under consideration here, self-collisions conserve mass, momentum, and energy in each gas, and therefore the three moments of the self-collision term on the right hand side of Eq. (5) vanish.

We generalize here the procedure which is used in the case of a single fluid [12], and seek an approximate solution for the distribution functions f_1 and f_2 . We examine Eq. (5) assuming that the system is collision dominated, and that the

cross section for self-collision is far greater than that of the intercollisions. To estimate the role of the different terms of Eq. (5) we introduce for each component three characteristic time scales: $\tau_{\alpha,\text{flow}}$ —the scale of hydrodynamic flow, $\tau_{\alpha\alpha}$ —the scale of self-collisions, and $\tau_{\alpha\beta}$ —the scale of intercollisions, as follows:

$$\frac{D}{Dt}f_\alpha \approx \frac{f_\alpha}{\tau_{\alpha,\text{flow}}}, \quad \mathcal{J}_{\alpha\alpha}[f_\alpha, f_\alpha] \approx \frac{f_\alpha}{\tau_{\alpha\alpha}}, \quad \mathcal{J}_{\alpha\beta}[f_\alpha, f_\beta] \approx \frac{f_\alpha}{\tau_{\alpha\beta}}. \quad (6)$$

The main assumption for our mixture is that $\mathcal{J}_{\alpha\alpha}$ is the dominant term in the kinetic equation for the α species, or that in terms of the time scales we have $\tau_{\alpha\alpha} \ll \tau_{\alpha,\text{flow}}$, and $\tau_{\alpha\alpha} \ll \tau_{\alpha\beta}$. We therefore expand

$$f_\alpha \approx f_\alpha^{(0)} + f_\alpha^{(1)}, \quad (7)$$

where

$$f_\alpha^{(1)} \approx \left(\frac{\tau_{\alpha\alpha}}{\tau_{\alpha,\text{flow}}} \right) f_\alpha^{(0)}, \quad \left(\frac{\tau_{\alpha\alpha}}{\tau_{\alpha\beta}} \right) f_\alpha^{(0)}, \quad (8)$$

i.e., $f_\alpha^{(1)} \ll f_\alpha^{(0)}$, and we can use perturbation theory. In the spirit of the Chapman-Enskog method we also demand that $f_\alpha^{(1)}$ in Eq. (7) has no contribution to the zeroth, first, and second velocity moments of f_α .

Since the dominant term of Eq. (5) is the first one on the right-hand side, the zero order term $f_\alpha^{(0)}$ is the solution of the equation

$$\mathcal{J}_{\alpha\alpha}[f_\alpha^{(0)}, f_\alpha^{(0)}] = 0. \quad (9)$$

The solution to Eq. (9) is simply the Maxwell-Boltzmann *local thermodynamic equilibrium* (LTE) distribution function, that is,

$$f_\alpha^{(0)}(\mathbf{r}, \mathbf{v}, t) = n_\alpha(\mathbf{r}, t) \left[\frac{m_\alpha}{2\pi\theta_\alpha(\mathbf{r}, t)} \right]^{3/2} \times \exp \left[- \frac{m_\alpha[\mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)]^2}{2\theta_\alpha(\mathbf{r}, t)} \right]. \quad (10)$$

Here $n_\alpha(\mathbf{r}, t)$, $\mathbf{u}_\alpha(\mathbf{r}, t)$, and $\theta_\alpha(\mathbf{r}, t)$ are the three *hydrodynamics variables* of the α component of the mixture: the local number density, average flow velocity, and kinetic temperature, respectively (see the Appendix for the proper definitions). Having the $f_\alpha^{(0)}$ expressed in terms of n_α , \mathbf{u}_α , and θ_α , we substitute Eq. (7) into Eq. (5) and find, to first order in the small dimensionless expansion parameters $(\tau_{\alpha\alpha}/\tau_{\alpha,\text{flow}})$ and $(\tau_{\alpha\alpha}/\tau_{\alpha\beta})$, an equation for $f_\alpha^{(1)}$:

$$\begin{aligned} \frac{D}{Dt}f_\alpha^{(0)}(\mathbf{r}, \mathbf{v}, t) &= \mathcal{J}_{\alpha\alpha}[f_\alpha^{(0)}, f_\alpha^{(1)}] + \mathcal{J}_{\alpha\alpha}[f_\alpha^{(1)}, f_\alpha^{(0)}] \\ &+ \mathcal{J}_{\alpha\beta}[f_\alpha^{(0)}, f_\beta^{(0)}]. \end{aligned} \quad (11)$$

These are implicit equations for $f_1^{(1)}$, and $f_2^{(1)}$, stated in terms of the six hydrodynamics variables, and the differential cross sections. In principle, once they are solved for $f_1^{(1)}$ and

$f_2^{(1)}$, we return to Eq. (5), substitute Eq. (7), and obtain the moment equations, i.e., the hydrodynamic equations for the mixture.

First we remark that, as for a single species, in the case of the binary mixture $f_\alpha^{(1)}$, the small correction to $f_\alpha^{(0)}$, contributes only to the stress tensor,

$$\begin{aligned} [\mathbf{\Pi}_\alpha(\mathbf{r}, t)]_{ij} &= m_\alpha \int d^3v [v_i - u_{\alpha,i}(\mathbf{r}, t)][v_j - u_{\alpha,j}(\mathbf{r}, t)] \\ &\times \{f_\alpha^{(0)}(\mathbf{r}, \mathbf{v}, t) + f_\alpha^{(1)}(\mathbf{r}, \mathbf{v}, t)\}, \end{aligned} \quad (12)$$

and to the thermal energy flux vector,

$$\mathbf{J}_\alpha(\mathbf{r}, t) = \frac{1}{2} m_\alpha^2 \int d^3v [\mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)][\mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)]^2 f_\alpha^{(1)}(\mathbf{r}, \mathbf{v}, t). \quad (13)$$

Here the indices i, j correspond to the three Cartesian vector components. Notice that for symmetry reasons the dominant part of the distribution function $f_\alpha^{(0)}(\mathbf{r}, \mathbf{v}, t)$ contributes only to the diagonal elements of the stress tensor ($[\mathbf{\Pi}_\alpha(\mathbf{r}, t)]_{ii}$ in Eq. (12), and does not contribute to its off-diagonal elements, nor to the thermal energy flux vector, Eq. (13). Then we observe that since the relevant moments of the first term, $\mathcal{J}_{\alpha\alpha}[f_\alpha, f_\alpha]$, on the right-hand side of Eq. (5) vanish, the only direct contributions to the hydrodynamics of the mixture, due to the elastic collisions, come from the moments of $\mathcal{J}_{\alpha\beta}[f_\alpha, f_\beta]$, the intercollision term. In the sense of our perturbation scheme, we end up with the following contributions of the moments:

$$M_\alpha^{(n)} = \int d^3v G_\alpha^{(n)}(\mathbf{v}) \mathcal{J}_{\alpha\beta}[f_\alpha^{(0)}, f_\beta^{(0)}], \quad (14)$$

where

$$G_\alpha^{(1)}(\mathbf{v}) = 1, \quad G_\alpha^{(2)}(\mathbf{v}) = m_\alpha \mathbf{v}, \quad G_\alpha^{(3)}(\mathbf{v}) = m_\alpha v^2/2. \quad (15)$$

III. CALCULATION OF THE INTERCOLLISION MOMENTS

We now wish to calculate the contribution of the intercollisions to the hydrodynamic equations of a gas in a binary mixture, e.g., the moments in Eqs. (14), (15). We focus on a mixture of atoms and ions in a plasma, in which the atomic masses of the two components are practically equal. The effect of intercollisions is expected to be especially significant in cases of large differences between the flow velocities and the kinetic temperatures of the two components. Since the case of similar (or identical) flow velocities with different temperatures was widely treated in the past (e.g., magneto-hydrodynamics in plasmas), we concentrate here on the case of large relative flow differences between the two gases. We assume that the velocity distributions of the two components are narrow and separated from each other, i.e., we introduce the small parameters:

$$\eta_\alpha \equiv \frac{v_\alpha^T}{|\mathbf{u}_\alpha - \mathbf{u}_\beta|} \ll 1, \quad (16)$$

where $v_\alpha^T = \sqrt{2\theta_\alpha/m}$ is the thermal velocity in component α , which represents also the width of its velocity distribution. In other words, we focus on the case of a large drift between the two components, which is physically meaningful in a variety of laboratory and natural plasmas. We will further assume that the thermal velocity of each of the two gases is small compared also to the other characteristic scales of the system, in velocity space.

We now use these assumptions to calculate the moments in Eqs. (14),(15). Examining Eq. (4) we see that each of the velocity moments in Eq. (14) is composed of two terms, $M_{\alpha,1}^{(n)}$ (the term with the primes in the $f^{(0)}$'s) and $M_{\alpha,2}^{(n)}$. The second term,

$$M_{\alpha,2}^{(n)} = - \int d^3v G_\alpha^{(n)}(\mathbf{v}) \int d^3v_1 d\Omega \sigma_{\alpha\beta}(\Omega) |\mathbf{v} - \mathbf{v}_1| f_\alpha^{(0)}(\mathbf{v}) f_\beta^{(0)}(\mathbf{v}_1), \quad (17)$$

can be cast into the form

$$M_{\alpha,2}^{(n)} = - \int d^3v G_\alpha^{(n)}(\mathbf{v}) \frac{f_\alpha^{(0)}(\mathbf{v})}{\tau_{\alpha\beta}(\mathbf{v})}, \quad (18)$$

where

$$\frac{1}{\tau_{\alpha\beta}(\mathbf{v})} \equiv \int d^3v_1 \int d\Omega \sigma_{\alpha\beta}(\Omega) |\mathbf{v} - \mathbf{v}_1| f_\beta^{(0)}(\mathbf{v}_1). \quad (19)$$

The physical meaning of $\tau_{\alpha\beta}(v)$ will be discussed later. We first observe that this can be written as

$$\frac{1}{\tau_{\alpha\beta}(\mathbf{v})} = n_\beta \Sigma_{\alpha\beta} v_\beta^T \frac{\Phi_2(-x) - \Phi_2(x)}{x}, \quad (20)$$

where $\Sigma_{\alpha\beta}$ is the total cross section of an intercollision encounter, with

$$x \equiv |\mathbf{v} - \mathbf{u}_\beta| / v_\beta^T, \quad (21)$$

and

$$\Phi_2(x) \equiv \int_x^\infty dx' \int_{x'}^\infty dx'' \operatorname{erfc}(x''). \quad (22)$$

To better understand the dependence of $\tau_{\alpha\beta}(\mathbf{v})$ on \mathbf{v} we approximate it as

$$\frac{1}{\tau_{\alpha\beta}(\mathbf{v})} = n_\beta \Sigma_{\alpha\beta} \left\{ \left[0.12 + \exp\left(-\frac{|\mathbf{v} - \mathbf{u}_\beta|}{v_\beta^T}\right) \right] v_\beta^T + \left| \mathbf{v} - \mathbf{u}_\beta \right| \right\}. \quad (23)$$

Figure 1 illustrates the functional dependence of $1/\tau_{\alpha\beta}(\mathbf{v})$ upon \mathbf{v} . The figure shows the exact expression, the approximate one, and the relative difference between them. We notice that the approximation of Eq. (23) is indeed within about one percent of Eq. (20). Note that $1/\tau_{\alpha\beta}(\mathbf{v})$ does not change

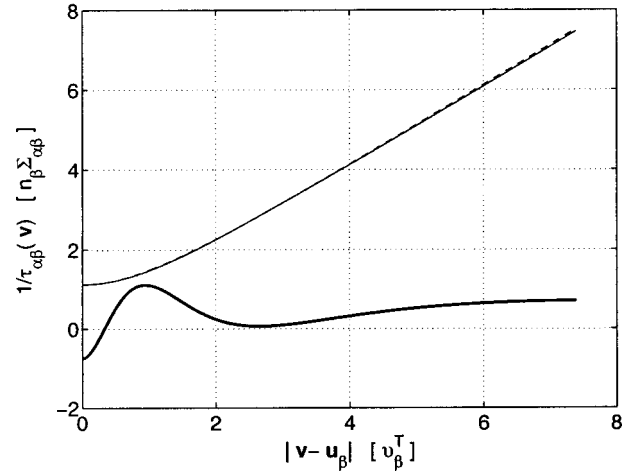


FIG. 1. The dependence of $1/\tau_{\alpha\beta}(\mathbf{v})$ upon $|\mathbf{v} - \mathbf{u}_\beta|$. $1/\tau_{\alpha\beta}(\mathbf{v})$ is given in units of $n_\beta \Sigma_{\alpha\beta}$ while $|\mathbf{v} - \mathbf{u}_\beta|$ is in units of v_β^T . Full thin line: the exact function of Eq. (20). Dashed line: the approximated function of Eq. (23). Thick line: the relative difference (%).

much ($\sim 25\%$) inside a sphere of radius v_β^T around \mathbf{u}_β in velocity space. We will get back to this point later on.

We now return to Eq. (17) and observe that since $f_\alpha^{(0)}(\mathbf{v})$ is narrow, we may regard $\tau_{\alpha\beta}(\mathbf{v})$ as approximately constant under the integration, and replace it by $\tau_{\alpha\beta}(\mathbf{u}_\alpha)$. Hence

$$M_{\alpha,2}^{(n)} = - \frac{1}{\tau_{\alpha\beta}(\mathbf{u}_\alpha)} \int d^3v G_\alpha^{(n)}(\mathbf{v}) f_\alpha(\mathbf{v}), \quad (24)$$

and $M_{\alpha,2}^{(n)}$ is proportional to the regular moments of $f_\alpha(\mathbf{v})$ (see the Appendix).

We turn now to the first term in Eqs. (14) $M_{\alpha,1}^{(n)}$. Using Eq. (3), adapting it to the case of equal atomic masses, we may write it as

$$M_{\alpha,1}^{(n)} = \int d^3v \int d^3v_1 \int d^3v' \int d^3v'_1 \Gamma(\mathbf{v}, \mathbf{v}_1 \rightarrow \mathbf{v}', \mathbf{v}'_1) \delta^3(\mathbf{v} + \mathbf{v}_1 - \mathbf{v}' - \mathbf{v}'_1) \delta(v^2 + v_1^2 - v'^2 - v'_1{}^2) \times G_\alpha^{(n)}(\mathbf{v}) f_\alpha^{(0)}(\mathbf{v}') f_\beta^{(0)}(\mathbf{v}'_1). \quad (25)$$

Since the functions $f_\alpha^{(0)}(\mathbf{v}')$ and $f_\beta^{(0)}(\mathbf{v}'_1)$ are assumed to be separated, and narrow with respect to the characteristic scales of changes in velocity space, we can first carry the integration over \mathbf{v}' and \mathbf{v}'_1 to find that

$$M_{\alpha,1}^{(n)} = n_\alpha n_\beta \int d^3v \int d^3v_1 \Gamma(\mathbf{v}, \mathbf{v}_1 \rightarrow \mathbf{u}_\alpha, \mathbf{u}_\beta) \delta^3(\mathbf{v} + \mathbf{v}_1 - \mathbf{u}_\alpha - \mathbf{u}_\beta) \delta(v^2 + v_1^2 - u_\alpha^2 - u_\beta^2) G_\alpha^{(n)}(\mathbf{v}). \quad (26)$$

The intercollisions scattering potential can be approximated using a hard spheres model (“billiard balls”). Under this approximation

$$\Gamma(\mathbf{v}, \mathbf{v}_1 \rightarrow \mathbf{u}_\alpha, \mathbf{u}_\beta) = \Gamma(\mathbf{v}_1, \mathbf{v} \rightarrow \mathbf{u}_\alpha, \mathbf{u}_\beta) \quad (27)$$

(this is easily seen using geometrical considerations for collisions between two identical hard spheres), and we may therefore replace the last expression for $M_{\alpha,1}^{(n)}$ by

$$M_{\alpha,1}^{(n)} = n_{\alpha} n_{\beta} \int d^3 v \int d^3 v_1 \Gamma(\mathbf{v}, \mathbf{v}_1 \rightarrow \mathbf{u}_{\alpha}, \mathbf{u}_{\beta}) \delta^3(\mathbf{v} + \mathbf{v}_1 - \mathbf{u}_{\alpha} - \mathbf{u}_{\beta}) \delta(v^2 + v_1^2 - u_{\alpha}^2 - u_{\beta}^2) \frac{G_{\alpha}^{(n)}(\mathbf{v}) + G_{\beta}^{(n)}(\mathbf{v}_1)}{2}, \quad (28)$$

and obtain

$$M_{\alpha,1}^{(n)} = n_{\alpha} n_{\beta} \sum_{\alpha\beta} |\mathbf{u}_{\alpha} - \mathbf{u}_{\beta}| \frac{G_{\alpha}^{(n)}(\mathbf{u}_{\alpha}) + G_{\beta}^{(n)}(\mathbf{u}_{\beta})}{2} \approx \frac{n_{\alpha}}{\tau_{\alpha\beta}(\mathbf{u}_{\alpha})} \frac{G_{\alpha}^{(n)}(\mathbf{u}_{\alpha}) + G_{\beta}^{(n)}(\mathbf{u}_{\beta})}{2}. \quad (29)$$

Combining Eqs. (24),(29) we finally find, for the zero moment,

$$M_{\alpha}^{(1)} = 0, \quad (30)$$

for the first moment,

$$M_{\alpha}^{(2)} = -m n_{\alpha} n_{\beta} \sum_{\alpha\beta} |\mathbf{u}_{\alpha} - \mathbf{u}_{\beta}| \frac{\mathbf{u}_{\alpha} - \mathbf{u}_{\beta}}{2} \approx -m \nu_{\alpha\beta}(\mathbf{u}_{\alpha}) \frac{\mathbf{u}_{\alpha} - \mathbf{u}_{\beta}}{2}, \quad (31)$$

and for the second moment,

$$M_{\alpha}^{(3)} = -\nu_{\alpha\beta}(\mathbf{u}_{\alpha}) \left[m \frac{\mathbf{u}_{\alpha}^2 - \mathbf{u}_{\beta}^2}{4} + \frac{3}{2} \theta_{\alpha} \right], \quad (32)$$

where $\nu_{\alpha\beta}(\mathbf{u}_{\alpha}) \equiv n_{\alpha} / \tau_{\alpha\beta}(\mathbf{u}_{\alpha})$ is the number of intercollisions between α and β per unit time per unit volume. Note that for very small η 's $\nu_{\alpha\beta}(\mathbf{u}_{\alpha}) \approx \nu_{\beta\alpha}(\mathbf{u}_{\beta}) (\equiv \nu)$, as expected.

These moments, $M_{\alpha}^{(1)}$, $M_{\alpha}^{(2)}$, and $M_{\alpha}^{(3)}$, are the contributions due to the ‘‘weak’’ inter-collisions to the flow equations of gas α in a mixture in the present model. These terms describe the exchange of mass, momentum, and energy (respectively) between the two mixed gases, due to intercollisions. We expect that intercollisions do not change the local *individual* mass of each component, nor the local *total* momentum and energy of the mixture. The conservation of each component’s mass is indeed demonstrated by Eq. (30), i.e., $M_{\alpha}^{(1)} = 0$. The conservation of total local momentum is evident using Eq. (31), since $M_{\alpha}^{(2)} + M_{\beta}^{(2)} = 0$. In contrast, the total local energy is not exactly conserved, since $M_{\alpha}^{(3)} + M_{\beta}^{(3)} = -\frac{3}{2} \nu (\theta_{\alpha} + \theta_{\beta})$. This is not surprising, since we have practically neglected the individual kinetic temperatures of the two components in the mixture. In the sense of the approximation given by Eq. (16), the error in the total local energy conservation is of first order in the η 's of Eq. (16). In the next section we will obtain first order corrections to Eqs. (30)–(32).

IV. RELAXATION TIME APPROXIMATION AND FIRST ORDER CORRECTIONS

The results obtained in the previous section, Eqs. (30)–(32), indicate that the effect of the intercollisions is to bring the dynamically separated two components of the binary mixture to relax towards a common flow velocity and kinetic temperature. This suggests that the intercollisions part of the Boltzmann relaxation term can be accounted for using the method of relaxation time approximation. This method is a useful tool in plasmas, and can give us here a better insight to the results of the previous section, and even help in modifying them to include first order corrections in η_{α} , η_{β} .

First let us recall the use of the relaxation time approximation in the case of one-component fluid, say α , where only self-collisions are under considerations. In the *single-parameter* relaxation time approximation the right hand side of Eq. (1), e.g., the SC collision operator, is replaced by

$$\mathcal{J}_{\alpha\alpha}[f_{\alpha}, f_{\alpha}] \rightarrow -\frac{f_{\alpha}(\mathbf{r}, \mathbf{v}, t) - f_{\alpha}^0(\mathbf{r}, \mathbf{v}, t)}{\tau_{\alpha\alpha}}. \quad (33)$$

Here the effect of self-collisions is modeled as a relaxation process, where f_{α} is approaching the LTE distribution over a time scale $\tau_{\alpha\alpha}$. The stress tensor $\mathbf{\Pi}_{\alpha}$ and thermal energy flux vector \mathbf{J}_{α} are then being calculated using perturbation theory for f_{α} around f_{α}^0 [12]. We first attempt to motivate Eq. (33), using the Boltzmann-collision operator of Eq. (4) for a single species.

Again we notice that the expression for $\mathcal{J}_{\alpha\alpha}$ in Eq. (4) is divided into two parts. The second one, namely,

$$\mathcal{J}_{\alpha\alpha}^{(2)}[f_{\alpha}, f_{\alpha}] = -\int d^3 v_1 d\Omega \sigma_{\alpha\alpha}(\Omega) |\mathbf{v} - \mathbf{v}_1| f_{\alpha}(\mathbf{v}) f_{\alpha}(\mathbf{v}_1), \quad (34)$$

can be clearly expressed as

$$\mathcal{J}_{\alpha\alpha}^{(2)}[f_{\alpha}, f_{\alpha}] = -\frac{f_{\alpha}(\mathbf{v})}{\tau_{\alpha\alpha}(\mathbf{v})}, \quad (35)$$

just by integrating over \mathbf{v}_1 . Since within the integral over \mathbf{v}_1 the distribution function $f_{\alpha}(\mathbf{v}_1)$ can be approximated by $f_{\alpha}^{(0)}(\mathbf{v}_1)$, we get for the velocity dependent relaxation time

$$\frac{1}{\tau_{\alpha\alpha}(\mathbf{v})} = \int d^3 v_1 d\Omega \sigma_{\alpha\alpha}(\Omega) |\mathbf{v} - \mathbf{v}_1| f_{\alpha}^{(0)}(\mathbf{v}_1), \quad (36)$$

which is equivalent to Eq. (19). The first term of Eq. (4),

$$\mathcal{J}_{\alpha\alpha}^{(1)}[f_{\alpha}, f_{\alpha}] = \int d^3 v_1 d\Omega \sigma_{\alpha\alpha}(\Omega) |\mathbf{v} - \mathbf{v}_1| f_{\alpha}(\mathbf{v}') f_{\alpha}(\mathbf{v}'_1), \quad (37)$$

needs more attention. Since the primed velocities are the products of the binary collisions we again can replace the distribution functions by their zero order expressions, i.e., f_{α} is replaced by $f_{\alpha}^{(0)}$. Furthermore, the conservation laws allow the replacement $f_{\alpha}^{(0)}(\mathbf{v}') f_{\alpha}^{(0)}(\mathbf{v}'_1) \Rightarrow f_{\alpha}^{(0)}(\mathbf{v}) f_{\alpha}^{(0)}(\mathbf{v}_1)$ and Eq. (37) can be written as

$$\begin{aligned} \mathcal{J}_{\alpha\alpha}^{(1)}[f_\alpha, f_\alpha] &= \int d^3v_1 d\Omega \sigma_{\alpha\alpha}(\Omega) |\mathbf{v} - \mathbf{v}_1| f_\alpha^{(0)}(\mathbf{v}) f_\alpha^{(0)}(\mathbf{v}_1) \\ &= \frac{f_\alpha^{(0)}(\mathbf{v})}{\tau_{\alpha\alpha}(\mathbf{v})}, \end{aligned}$$

with the same $\tau_{\alpha\alpha}(\mathbf{v})$ of Eq. (36), and thus

$$\mathcal{J}_{\alpha\alpha}[f_\alpha, f_\alpha] \approx -\frac{f_\alpha(\mathbf{r}, \mathbf{v}, t) - f_\alpha^0(\mathbf{r}, \mathbf{v}, t)}{\tau_{\alpha\alpha}(\mathbf{v})}. \quad (38)$$

This justifies Eq. (33), however, with a velocity-dependent time scale, $\tau_{\alpha\alpha}(\mathbf{v})$. Figure 1 shows the dependence of this time scale on \mathbf{v} (simply change the subscripts β to α in the figure caption and axis). It is easy to see that $\tau_{\alpha\alpha}(\mathbf{v})$ changes very little (by about a quarter) in a sphere of radius v_α^T around \mathbf{u}_α . In velocity space most of the particles of the gas are located inside this sphere, and therefore using a constant relaxation time scale seems fairly appropriate.

We now wish to extend the method of relaxation time approximation (RTA) to the case of a binary gas mixture, in a manner that would result similar contributions, due to intercollisions, to the flow equations, as were obtained in the previous section. We expect the relaxation term in Eq. (33), due to the intercollisions, to have a form analogous to that of Eq. (2), i.e.,

$$\mathcal{J}_{\alpha\beta}[f_\alpha, f_\beta] \rightarrow -\frac{f_\alpha(\mathbf{r}, \mathbf{v}, t) - F_{\alpha\beta}(\mathbf{r}, \mathbf{v}, t)}{\tau_{\alpha\beta}}, \quad (39)$$

where

$$\begin{aligned} F_{\alpha\beta}(\mathbf{r}, \mathbf{v}, t) &= n_\alpha(\mathbf{r}, t) \left[\frac{m_\alpha}{2\pi\Theta_{\alpha\beta}(\mathbf{r}, t)} \right]^{3/2} \\ &\times \exp \left[-\frac{m_\alpha[\mathbf{v} - \mathbf{U}_{\alpha\beta}(\mathbf{r}, t)]^2}{2\Theta_{\alpha\beta}(\mathbf{r}, t)} \right], \end{aligned} \quad (40)$$

and

$$\mathbf{U}_{\alpha\beta} = \mathbf{U}_{\beta\alpha}, \quad \Theta_{\alpha\beta} = \Theta_{\beta\alpha}. \quad (41)$$

Here the net effect of intercollisions is now assumed to make f_α relax towards a ‘‘common’’ LTE state $F_{\alpha\beta}$ on a time scale of $\tau_{\alpha\beta}$ [14]. The parameters of the common LTE (average velocity and width of the distribution) are different from the ‘‘self-LTE’’ parameters of f_α^0 . Equation (39) demands that the first and second velocity moments of $F_{\alpha\beta}$ (the flow velocity and kinetic temperature, respectively) are equal to those of $F_{\beta\alpha}$ (to which the intercollisions force f_β to relax). This expresses the tendency of the intercollisions to equate the flow velocity and kinetic temperature of the two species..

Using Eqs. (1),(2),(33),(39) we get the equations of motion for the distribution functions of two mixed gases in the RTA scheme:

$$\frac{D}{Dt} f_\alpha = -\frac{f_\alpha(\mathbf{r}, \mathbf{v}, t) - f_\alpha^0(\mathbf{r}, \mathbf{v}, t)}{\tau_\alpha} - \frac{f_\alpha(\mathbf{r}, \mathbf{v}, t) - F_{\alpha\beta}(\mathbf{r}, \mathbf{v}, t)}{\tau_{\alpha\beta}}. \quad (42)$$

The parameters of $F_{\alpha\beta}$ in Eq. (39), $\mathbf{U}_{\alpha\beta}$ and $\Theta_{\alpha\beta}$, are functions of the individual flow parameters of both components of the mixture. The velocity moments of Eq. (42) determine the intercollisions contributions to the flow equations in the RTA approximation. We compare these contributions to those obtained at the end of the last section, Eqs. (30)–(32). It is easy to check that choosing

$$\mathbf{U}_{\alpha\beta}(\mathbf{r}, t) = \frac{u_\alpha(\mathbf{r}, t) + u_\beta(\mathbf{r}, t)}{2}, \quad (43)$$

$$\Theta_{\alpha\beta}(\mathbf{r}, t) = \frac{m}{12} [u_\alpha(\mathbf{r}, t) - u_\beta(\mathbf{r}, t)]^2, \quad (44)$$

and

$$\frac{1}{\tau_{\alpha\beta}} = n_\beta \Sigma_{\alpha\beta} |\mathbf{u}_\alpha - \mathbf{u}_\beta|, \quad (45)$$

yields the desired result. Notice that the expressions for $\Theta_{\alpha\beta}$ and $\tau_{\alpha\beta}$ are independent of the individual kinetic temperatures of the two gases θ_α and θ_β . This is not surprising since $\mathbf{U}_{\alpha\beta}(\mathbf{r}, t)$, $\Theta_{\alpha\beta}(\mathbf{r}, t)$, and $\tau_{\alpha\beta}(\mathbf{r}, t)$ were chosen to be applicable in the limit where η_α and η_β are very small, Eq. (16), which assumes that the individual kinetic temperatures of the gases are small, in some sense. Equations (30)–(32) include only zero order terms in η_1 and η_2 , and correspondingly also $\mathbf{U}_{\alpha\beta}(\mathbf{r}, t)$, $\Theta_{\alpha\beta}(\mathbf{r}, t)$, and $\tau_{\alpha\beta}(\mathbf{r}, t)$ of Eqs. (43)–(45). We now wish to use the RTA scheme to estimate the first order corrections in η_α and η_β to the parameters $\Theta_{\alpha\beta}(\mathbf{r}, t)$ and $\tau_{\alpha\beta}(\mathbf{r}, t)$.

(i) *The corrections to $\Theta_{\alpha\beta}$.* We have stated before that $M_\alpha^{(2)}$, which describes the exchange of energy between the two gases due to intercollisions, conserves the total local energy only to zero order in η_α and η_β . We now suggest a first order correction to the common kinetic temperature $\Theta_{\alpha\beta}$. Writing $\Theta_{\alpha\beta} \rightarrow \Theta_{\alpha\beta}^* = \Theta_{\alpha\beta} + \Delta\Theta$, where $\Delta\Theta/\Theta_{\alpha\beta} \sim \eta_\alpha, \eta_\beta$, we examine the energy moment ($\int d^3v \frac{1}{2} m_\alpha v^2$) of Eq. (42). The energy continuity equation is written as

$$\begin{aligned} \frac{\partial}{\partial t} (\epsilon_\alpha) + \nabla \cdot (\epsilon_\alpha \mathbf{u}_\alpha) + \nabla \cdot \mathbf{J}_\alpha + \nabla \cdot (\mathbf{\Pi}_\alpha \cdot \mathbf{u}_\alpha) - n_\alpha \mathbf{F}_\alpha \cdot \mathbf{u}_\alpha \\ = \frac{1}{\tau_{\alpha\beta}} \left[\frac{3}{2} n_\alpha \Theta_{\alpha\beta}^* + \frac{1}{2} m_\alpha n_\alpha \mathbf{U}_{\alpha\beta}^2 - \frac{3}{2} n_\alpha \theta_\alpha + \frac{1}{2} m_\alpha n_\alpha \mathbf{u}_\alpha^2 \right]. \end{aligned} \quad (46)$$

In order to have total local energy of the mixture conserved, we demand that the sum of the right hand sides of these equations, over the two fluids should vanish, i.e.,

$$\begin{aligned} \frac{1}{\tau_{\alpha\beta}} \left[\frac{3}{2} n_\alpha \Theta_{\alpha\beta}^* + \frac{1}{2} m n_\alpha \mathbf{U}_{\alpha\beta}^2 - \left(\frac{3}{2} n_\alpha \theta_\alpha + \frac{1}{2} m n_\alpha \mathbf{u}_\alpha^2 \right) \right] \\ + \frac{1}{\tau_{\beta\alpha}} \left[\frac{3}{2} n_\beta \Theta_{\alpha\beta}^* + \frac{1}{2} m n_\beta \mathbf{U}_{\alpha\beta}^2 - \left(\frac{3}{2} n_\beta \theta_\beta + \frac{1}{2} m n_\beta \mathbf{u}_\beta^2 \right) \right] \\ = 0. \end{aligned} \quad (47)$$

Since, by Eq. (41), $\Theta_{\alpha\beta}^* = \Theta_{\beta\alpha}^*$, $\mathbf{U}_{\alpha\beta} = \mathbf{U}_{\beta\alpha}$ and, by Eq. (45), $n_\alpha/\tau_{\alpha\beta} = n_\beta/\tau_{\beta\alpha}$, and substituting $\Theta_{\alpha\beta}^* = \Theta_{\alpha\beta} + \Delta\Theta$, we find that the last expression reduces into

$$6(\Theta + \Delta\Theta) + 2m\mathbf{U}^2 - 3(\theta_\alpha + \theta_\beta) - m(\mathbf{u}_\alpha^2 + \mathbf{u}_\beta^2) = 0, \quad (48)$$

which is the desired equation for $\Delta\Theta$. Substituting Θ and \mathbf{U} from Eqs. (43),(44) we find that $\Delta\Theta = (\theta_\alpha + \theta_\beta)/2$, which is indeed a first order correction to $\Theta_{\alpha\beta}$, in η_α and η_β . In summary the common kinetic temperature up to first order in η_α and η_β is

$$\Theta_{\alpha\beta}(\mathbf{r}, t) = \frac{m}{12} [u_\alpha(\mathbf{r}, t) - u_\beta(\mathbf{r}, t)]^2 + \frac{\theta_\alpha + \theta_\beta}{2}. \quad (49)$$

(ii) *The corrections to $\tau_{\alpha\beta}$.* Equation (23) gives an approximate expression for the velocity-dependent relaxation time $\tau_{\alpha\beta}(\mathbf{v})$. It represents the effect of many intercollisions encounters on a particle of component α , having a specific velocity \mathbf{v} . It is reasonable to suggest that the *constant* relaxation time $\tau_{\alpha\beta}$ should be the average of $\tau_{\alpha\beta}(\mathbf{v})$ over *all* the particles of component α :

$$\frac{1}{\tau_{\alpha\beta}(\mathbf{r}, t)} = \int d^3v f_\alpha(\mathbf{r}, t) \frac{1}{\tau_{\alpha\beta}(r, \mathbf{v}, t)}. \quad (50)$$

Again, we can use Eq. (7) to replace f_α by $f_\alpha^{(0)}$ in the integration. We now use the approximate expression of $\tau_{\alpha\beta}(\mathbf{v})$ in Eq. (23) to get $\tau_{\alpha\beta}$, in the limit of Eq. (16). Neglecting second order effect in η_α and η_β we get that

$$\frac{1}{\tau_{\alpha\beta}} = n_\beta \Sigma_{\alpha\beta} [|\mathbf{u}_\alpha - \mathbf{u}_\beta| + 0.12(v_\alpha^T + v_\beta^T)]. \quad (51)$$

Indeed we see that this expression for $\tau_{\alpha\beta}$ is a sum of the zero order term, which was already obtained in Eq. (45), and a first order correction in η_α and η_β .

To summarize, we have developed a simple RTA relaxation term, given by Eqs. (42), for the dynamics of two, collisional coupled mixed gases. We have also calculated the appropriate parameters, given by Eqs. (43),(49),(51), in this RTA scheme. This, in turn, reproduces the proper hydrodynamic equations for a binary mixture, in the important case where a significant flow velocity difference is maintained between the two components of the fluid (i.e., with no significant overlap between the two distribution functions).

The RTA approach interprets the effect of collisions as a combination of two different relaxation processes: (i) Collisions between the same particles, e.g., of fluid α , which tend to relax this fluid into its own LTE state on a time scale $\tau_{\alpha\alpha}$. This is the faster of the two processes. (ii) Intercollisions between particles of fluids α and β , which tend to relax each fluid from its own LTE towards a common LTE, on a time scale of $\tau_{\alpha\beta}$. In this common LTE the two fluids have the same flow velocity and kinetic temperature. Note that Eq. (51) yields $\tau_{\alpha\beta} \neq \tau_{\beta\alpha}$. Actually we see that the (local) ‘‘rate’’ of relaxation of each gas is proportional to the local density of the other gas. Defining

$$\nu_{\alpha\beta} = n_\alpha/\tau_{\alpha\beta} = n_\alpha n_\beta \Sigma_{\alpha\beta} [|\mathbf{u}_\alpha - \mathbf{u}_\beta| + 0.12(v_\alpha^T + v_\beta^T)] \quad (52)$$

as the number of collisions of particle of species α with particles of species β , per unit time per unit volume, we get $\nu_{\alpha\beta} = \nu_{\beta\alpha}$, as expected.

Before we use the suggested kinetic Eqs. (39)-(51) to derive the resulting moments equations, we like to offer a qualitative description of the expected dynamics of the system. We examine the case for which the two fluids in the mixture have separate flow velocities and kinetic temperatures, that is when the coupling effect of interspecies collisions is weaker than the effect of self-collisions. We then expect to have $\tau_{11}, \tau_{22} \ll \tau_{12}, \tau_{21}$, and thus expect the distribution functions f_1, f_2 to first relax on the short time scales, τ_{11}, τ_{22} , respectively, to LTE forms, like that of Eq. (10), with different parameters of flow velocity ($\mathbf{u}_1 \neq \mathbf{u}_2$) and kinetic temperature ($\theta_1 \neq \theta_2$). However, on the long time scales, τ_{12}, τ_{21} , respectively, we expect the characteristic parameters of both LTE distributions, \mathbf{u}_1, θ_1 and \mathbf{u}_2, θ_2 , to become \mathbf{U}, Θ , respectively (where we now omit the indices 12 from the *common* velocity and kinetic temperature).

We are now ready to summarize our analysis of the binary mixture. Using the results for the relaxation terms, Eqs. (39)-(51), and taking the moments of the kinetic equations for f_α and f_β , we write down the set of hydrodynamic flow equations for the mixture as follows.

(1) *Conservation of mass,*

$$\frac{\partial}{\partial t} m_\alpha n_\alpha + \nabla \cdot (m_\alpha n_\alpha \mathbf{u}_\alpha) = \mathcal{S}^{\alpha\beta}, \quad (53)$$

where $\mathcal{S}^{\alpha\beta}$ on the right-hand side represents the sources of the masses. Since the intercollision encounters do not involve mass flux or exchange between the two fluids, we get

$$\mathcal{S}_{ic}^{\alpha\beta} = 0. \quad (54)$$

(2) *Conservation of momentum,*

$$\frac{\partial}{\partial t} (m_\alpha n_\alpha \mathbf{u}_\alpha) + \nabla \cdot \mathbf{u}_\alpha (m_\alpha n_\alpha \mathbf{u}_\alpha) + \nabla \cdot \mathbf{\Pi}_\alpha - n_\alpha \mathbf{F}_\alpha = \mathcal{F}^{\alpha\beta}, \quad (55)$$

where, here, $\mathcal{F}^{\alpha\beta}$ corresponds to momentum sources, and equals

$$\mathcal{F}_{ic}^{\alpha\beta} = m_\alpha \nu_{\alpha\beta} \frac{(\mathbf{u}_\alpha - \mathbf{u}_\beta)}{2}. \quad (56)$$

(3) *Conservation of energy,*

$$\frac{\partial}{\partial t} \epsilon_\alpha + \nabla \cdot \mathbf{u}_\alpha \epsilon_\alpha + \nabla \cdot \mathbf{J}_\alpha + \nabla \cdot (\mathbf{\Pi}_\alpha \cdot \mathbf{u}_\alpha) - n_\alpha \mathbf{F}_\alpha \cdot \mathbf{u}_\alpha = \mathcal{H}^{\alpha\beta}, \quad (57)$$

where now $\mathcal{H}^{\alpha\beta}$ takes care of the energy density sources, and is

$$\begin{aligned} \mathcal{H}_{ic}^{\alpha\beta} = & \nu_{\alpha\beta} \left[\frac{3}{4}(\theta_\beta - \theta_\alpha) + \frac{1}{2}m_\alpha(\mathbf{U}^2 - \mathbf{u}_\alpha^2) \right] \\ & + \nu_{\alpha\beta} \frac{m_\alpha}{8}(\mathbf{u}_\alpha - \mathbf{u}_\beta)^2. \end{aligned} \quad (58)$$

Note that $\nu_{\alpha\beta}$ from Eqs. (56),(58) is explicitly given above, in Eq. (52).

This is our general result for the flow equations of a mixture of two coupled gases. The collisional coupling between the two components of the mixture is expressed by the rate of change of momentum, $\mathcal{F}_{ic}^{\alpha\beta}$ of Eq. (56), and by the energy rate of change, $\mathcal{H}_{ic}^{\alpha\beta}$ of Eq. (58). $\mathcal{F}_{ic}^{\alpha\beta}$ represents the collisional drag which fluid β exerts on fluid α . $\mathcal{H}_{ic}^{\alpha\beta}$ describes the energy exchange between the two fluids due to the interaction between them: the first term indicates the tendency of the two components to reach a common temperature; the second term describes the relaxation of the kinetic energy to the center-of-mass kinetic energy of both fluids; and the third term comes from dissipation of kinetic energy into thermal energy. We can further simplify the expression for $\mathcal{H}_{ic}^{\alpha\beta}$ by substituting the expression for $\mathbf{U}_{\alpha\beta}$, Eq. (43):

$$\mathcal{H}_{ic}^{\alpha\beta} = \nu_{\alpha\beta} \left[\frac{3}{4}(\theta_\beta - \theta_\alpha) + \frac{1}{4}m_\alpha(\mathbf{u}_\beta^2 - \mathbf{u}_\alpha^2) \right]. \quad (59)$$

V. IONIZATION, RECOMBINATION, AND CHARGE EXCHANGE

Up to now our discussion was valid for *any* mixture of two collisional coupled gases, with equal or very nearly equal atomic masses, when a difference in both flow velocities and kinetic temperatures is maintained. In the case of partially ionized plasma, we have dealt only with one of the four coupling mechanism between the neutral-species gas (atoms) and the charged-species gas (ions and electrons) of which this ‘‘mixture’’ consists [13]. While intercollisions *exchange* momentum and energy between these two gases, the three additional coupling mechanisms—ionization, recombination, and charge exchange—exchange also *mass* between the atoms and the ions. We now suggest to investigate these additional coupling mechanisms, using a RTA scheme, similar to that of the elastic intercollisions. To carry on this program we reexamine Eqs. (42) in the presence of additional coupling mechanisms between the species. We shall use the indices a and i for atoms and ions, respectively. We return to Eq. (1) and recall that the influence of the close-range interactions between the particles, has been denoted by $(\partial f / \partial t)_{\text{coll}}$, and we write it now as

$$\left(\frac{\partial}{\partial t} f_\alpha \right)_{\text{coll}} = \left(\frac{\partial}{\partial t} f_\alpha \right)_{\text{sc}} + \left(\frac{\partial}{\partial t} f_\alpha \right)_{\text{ic}} + \left(\frac{\partial}{\partial t} f_\alpha \right)_{\text{ir}} + \left(\frac{\partial}{\partial t} f_\alpha \right)_{\text{cx}}. \quad (60)$$

Here the first term on the right-hand side stands for the influence of self-collisions (in fact all close-range interactions) inside each component in the mixture, the second term represents the influence of interspecies collisions, which were

discussed in previous sections, the third term stands for ionization-recombination, and the fourth term stands for charge exchange.

Before we continue to investigate the roles of charge exchange, ionization and recombination on the flow equations for plasmas, let us re-express the source terms due to atom-ion intercollisions. In the present notations we have

$$\begin{aligned} & \alpha, \quad \beta \rightarrow a, \quad i, \\ & \tau_{\alpha\alpha}, \quad \tau_{\alpha\beta}, \quad \nu_{\alpha\beta} \rightarrow \tau_{\text{sc}}^a, \quad \tau_{\text{ic}}^{ai}, \quad \nu_{\text{ic}}^{ai}, \\ & \mathcal{S}^{\alpha\beta}, \quad \mathcal{F}^{\alpha\beta}, \quad \mathcal{H}^{\alpha\beta} \rightarrow \mathcal{S}_{\text{ic}}^{ai}, \quad \mathcal{F}_{\text{ic}}^{ai}, \quad \mathcal{H}_{\text{ic}}^{ai} \end{aligned}$$

(swap a and i to get the expressions in the ions flow equations). For example, the expression in Eq. (56), for the effective drag exerted on the atoms by the ions, is now written as

$$\mathcal{F}_{\text{ic}}^{ai} = -m \nu_{\text{ic}} \frac{\mathbf{u}_a - \mathbf{u}_i}{2}. \quad (61)$$

We start with the rate of change of the distribution functions due to ionization and recombination (IR). In analogy with Eq. (39) we take, as was also suggested by Catto [3] and Hazeltine *et al.* [15],

$$\begin{aligned} \left(\frac{\partial}{\partial t} f_a \right)_{\text{ir}} &= -\frac{f_a}{\tau_{\text{ion}}} + \frac{f_i}{\tau_{\text{rec}}}, \\ \left(\frac{\partial}{\partial t} f_i \right)_{\text{ir}} &= +\frac{f_a}{\tau_{\text{ion}}} - \frac{f_i}{\tau_{\text{rec}}}, \end{aligned} \quad (62)$$

where τ_{ion} is the mean ‘‘lifetime’’ of an atom in the plasma before it is ionized, and τ_{rec} is the mean lifetime of an ion before a recombination with an electron. Examining Eqs. (62) we clearly recognize that the terms describing ionization and recombination processes introduce exchange of mass, momentum, and energy between the atoms and the ions, though conservation of the total values of these quantities in the plasma is maintained by these equations. Taking the moments of Eqs. (62) additional terms should be added to Eqs. (53),(55),(57). Defining the ionization and recombination rates as $\nu_{\text{ion}} = n_a / \tau_{\text{ion}}$ and $\nu_{\text{rec}} = n_i / \tau_{\text{rec}}$, we get that the rate of mass transfer from the ions to the atoms, due to the IR processes, is

$$\mathcal{S}_{\text{ir}}^{ai} = -\nu_{\text{ion}} m_a + \nu_{\text{rec}} m_i; \quad (63)$$

the effective drag force exerted by the ions on the atoms is

$$\mathcal{F}_{\text{ir}}^{ai} = -\nu_{\text{ion}} m_a \mathbf{u}_a + \nu_{\text{rec}} m_i \mathbf{u}_i; \quad (64)$$

and, finally, the energy transfer rate from the ions to the atoms, by IR, is

$$\mathcal{H}_{\text{ir}}^{ai} = -\nu_{\text{ion}} \frac{1}{2} m_a \mathbf{u}_a^2 + \nu_{\text{rec}} \frac{1}{2} m_i \mathbf{u}_i^2 \quad (65)$$

in the atoms “energy” equation. The same terms now appear also in the ions-gas mass, momentum and energy equations, however, with opposite signs. Note that the mass transferred in a single ionization or recombination event is always m_a .

We then turn to the last term of Eq. (60), and study the effect of charge exchange on the source terms of Eqs. (53),(55),(57). Using Eqs. (2),(4) we find the Boltzmann collision operator for charge exchange reactions as

$$\left(\frac{\partial}{\partial t} f_i\right)_{\text{cx}} = \int d^3 v_1 \sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) |\mathbf{v}-\mathbf{v}_1| [f_i(\mathbf{v}_1) f_a(\mathbf{v}) - f_i(\mathbf{v}) f_a(\mathbf{v}_1)] \quad (66)$$

(swap a and i to get the expression in the atoms kinetic equation). Note that in a charge exchange reaction $\mathbf{v}, \mathbf{v}_1 \rightarrow \mathbf{v}_1, \mathbf{v}$, and therefore we have used $\mathbf{v}', \mathbf{v}'_1 = \mathbf{v}_1, \mathbf{v}$ to get the last expression.

The cross section for a charge exchange reaction decreases with increasing values of the impact velocity [16,17]. The general approach to incorporate charge exchange processes in the kinetic level is to assume that $\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) \cdot |\mathbf{v}-\mathbf{v}_1|$ varies slowly in velocity space, and to neglect this variation [2,3,15]. This conjecture simplifies the Boltzmann collision operator for the charge exchange and allows an easy derivation of its velocity moments, e.g., its contribution to the flow equations.

The velocity dependence of σ_{cx} on $|\mathbf{v}-\mathbf{v}_1|$ is *not*, however, an inverse-proportional rule. The well known approximate relation for a *symmetric resonant* charge exchange [18],

$$\sqrt{\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|)} = -k_1 \ln(|\mathbf{v}-\mathbf{v}_1|) + k_2, \quad (67)$$

was theoretically suggested previously [16], and confirmed in experiments [17]. It is therefore clear that neglecting the variation of $\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) |\mathbf{v}-\mathbf{v}_1|$ is appropriate only for a very small range of impact velocities (or energies). Alas, in a mixture of atoms and ions (a partially ionized plasma) the range of impact velocities is not small enough for $\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) |\mathbf{v}-\mathbf{v}_1|$ to be constant. Atoms and ions coexist together in a plasma up to temperatures of 2–3 eV s. Accordingly, even without relative flow between the two species, $\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) |\mathbf{v}-\mathbf{v}_1|$ changes a lot. For example, in Ar plasma $\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) \cdot |\mathbf{v}-\mathbf{v}_1|$ changes by 100% or so, in the range of 1–3 eV. When a significant relative flow is also considered, the changes may be more than an order of magnitude. We conclude that the traditional approach to deal with charge exchange is not adequate for the case of large relative flow between the atoms and the ions, in which we are interested.

We want to find the contribution of charge exchange reactions to the flow equations of mixed atoms and ions, without assuming the inverse-proportional law of cross section. The formalism that we have developed in the previous sections allows it, especially when a large relative flow is developed. As in Eqs. (14),(15), the contributions of charge exchange to the flow equations are the following velocity moments of the collision operator:

$$M_i^{(n),\text{cx}} = \int d^3 v G^{(n)}(\mathbf{v}) \int d^3 v_1 \sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) |\mathbf{v} - \mathbf{v}_1| [f_i(\mathbf{v}_1) f_a(\mathbf{v}) - f_i(\mathbf{v}) f_a(\mathbf{v}_1)], \quad (68)$$

where $G^{(n)}(\mathbf{v})$ is defined as in Eq. (15). The last expression is easily cast into the symmetrical form

$$M_i^{(n),\text{cx}} = \int d^3 v d^3 v_1 \frac{G^{(n)}(\mathbf{v}) - G^{(n)}(\mathbf{v}_1)}{2} \sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|) |\mathbf{v} - \mathbf{v}_1| [f_i(\mathbf{v}_1) f_a(\mathbf{v}) - f_i(\mathbf{v}) f_a(\mathbf{v}_1)]. \quad (69)$$

The case of large relative flow is expressed by Eq. (16), which means that the distribution functions of the atoms and the ions are narrow and separated. Under such conditions, the value of $\sigma_{\text{cx}}(|\mathbf{v}-\mathbf{v}_1|)$ —for example, as given in Eq. (67)—is almost constant under the velocity integrations in Eq. (69). Similarly, also the other variables of the f 's inside these velocity moments are almost fixed. Therefore, to zero order in the η 's of Eq. (16), the velocity moments of the charge exchange collision integral in the ions kinetic equation are

$$M_i^{(1),\text{cx}} = 0, \quad (70)$$

$$M_i^{(2),\text{cx}} = -m \nu_{\text{cx}} (\mathbf{u}_i - \mathbf{u}_a), \quad (71)$$

$$M_i^{(3),\text{cx}} = -m \nu_{\text{cx}} \frac{\mathbf{u}_i^2 - \mathbf{u}_a^2}{2}, \quad (72)$$

where

$$\nu_{\text{cx}} = n_i n_a |\mathbf{u}_i - \mathbf{u}_a| \sigma_{\text{cx}}(|\mathbf{u}_i - \mathbf{u}_a|) \quad (73)$$

[swap the indices a and i to get the equivalent of Eqs. (70)–(73) for the atoms flow equations]. Here ν_{cx} is the rate of atom-ion charge exchange reactions in a unit volume.

By now we have obtained zero order expressions (in η_a, η_i) for the contribution of charge exchange to the flow equations. From our treatment in elastic collisions we remember that first order corrections for the velocity moments were obtained using a relaxation time approximation (see Sec. IV). To get first order corrections to Eqs. (70)–(73) we suggest the following RTA model:

$$\left(\frac{\partial}{\partial t} f_a\right)_{\text{cx}} = -\frac{f_a - \lambda_{ai} f_i}{\tau_{\text{cx}}^{ai}},$$

$$\left(\frac{\partial}{\partial t} f_i\right)_{\text{cx}} = -\frac{f_i - \lambda_{ia} f_a}{\tau_{\text{cx}}^{ia}}, \quad (74)$$

with

$$\tau_{\text{cx}}^{ai} = n_a / \nu_{\text{cx}}, \quad \tau_{\text{cx}}^{ia} = n_i / \nu_{\text{cx}}. \quad (75)$$

τ_{cx}^{ai} is the average lifetime of an atom before it undergoes a charge exchange reaction and becomes an ion, while τ_{cx}^{ia} is the average lifetime of an ion before it undergoes a reaction and transforms to be an atom, and $\lambda_{ai}, \lambda_{ia}$ are assumed to be

independent of velocity space. The meaning of the λ 's will shortly be revealed. First we want to give an intuitive interpretation to Eq. (74), and compare it with the RTA description of the elastic intercollisions in Eqs. (39)–(41). Back there our interpretation was that intercollisions encounter between the two gases relax both of them into a joint state, in which they have the common flow velocity and kinetic temperature. Our interpretation for charge exchange, however, is naturally different. Equation (74) mean that the net effect of many charge exchange reactions is to bring the velocity dependence of f_a directly to the that of f_i (and vice versa). This is quite reasonable since an atom-ion charge exchange reaction swap the charge of the two particles, without any change of their velocity or mass. A large number of charge exchange reactions therefore will make the atoms behave like ions in velocity space (and vice versa), while they conserve mass and particle density of each fluid. Note also that, exactly as the result for elastic intercollisions, the relaxation times of atoms and ions due to charge exchange are not equal, but rather related to each other like their number ratios.

Since locally the total rate of changes, due to charge exchange, is conserved, i.e., $(\partial f_a / \partial t)_{\text{cx}} + (\partial f_i / \partial t)_{\text{cx}} = 0$, we get from Eqs. (74)

$$f_a \left(\frac{1}{\tau_{\text{cx}}^{ai}} - \frac{\lambda_{ai}}{\tau_{\text{cx}}^{ia}} \right) + f_i \left(\frac{1}{\tau_{\text{cx}}^{ia}} - \frac{\lambda_{ia}}{\tau_{\text{cx}}^{ai}} \right) = 0. \quad (76)$$

In order for this to hold for all f_a and f_i , the bracketed expressions in this equation should vanish. We therefore conclude that

$$\lambda_{ai} = \frac{\tau_{\text{cx}}^{ai}}{\tau_{\text{cx}}^{ia}}, \quad \lambda_{ia} = \frac{\tau_{\text{cx}}^{ia}}{\tau_{\text{cx}}^{ai}} \left(\text{therefore } \lambda_{ai} = \frac{1}{\lambda_{ia}} \right), \quad (77)$$

or, using Eq. (75),

$$\lambda_{ai} = \frac{n_a}{n_i}, \quad \lambda_{ia} = \frac{n_i}{n_a}. \quad (78)$$

We see that the λ 's are practically weight factors which conserve the particle (and mass) density during charge exchange reactions in *each* fluid. Actually, it is easily seen that Eq. (78) also secures conservation of *total* momentum and energy of the mixture. For example, applying the first velocity moment on Eqs. (74) and using Eq. (78) we get

$$-\frac{m_a n_a (\mathbf{u}_a - \mathbf{u}_i)}{\tau_{\text{cx}}^{ai}} - \frac{m_a n_i (\mathbf{u}_i - \mathbf{u}_a)}{\tau_{\text{cx}}^{ia}} = 0. \quad (79)$$

The contribution of charge exchange to the drag exerted by the ions on the atoms is simply [compare this result with Eq. (61)]

$$\mathcal{F}_{\text{cx}}^{ai} = -m \nu_{\text{cx}} (\mathbf{u}_a - \mathbf{u}_i), \quad (80)$$

while exactly the opposite force exerted by the atoms on the ions. At this point it is interesting to discuss the difference between the drag force, $\mathcal{F}_{\text{ic}}^{ai}$ of Eq. (61), which is due to the intercollisions coupling, and $\mathcal{F}_{\text{cx}}^{ai}$ of Eq. (80), which is caused

by the charge exchange coupling. We observe that they do not differ only by their different rates, ν_{ic} and ν_{cx} , but also by a factor of 1/2. This means that *each* charge exchange reaction is twice “stronger” than the equivalent elastic intercollision. The result is intriguing, since it corresponds to a *microscopic* result. A charge exchange reaction between an atom and an ion is analog (in its results) to a *head-on* elastic intercollision. The momentum which is transferred in a *head-on* collision between two particles of the same mass, with definite velocities, is exactly twice the momentum transferred in an *average* elastic collision between two particles having the same definite velocities.

To get the contribution of charge exchange to the energy transfer between the two mixed components in the plasma we apply the second order velocity moment on Eqs. (74). The energy transferred from the ions to the atoms is

$$\mathcal{H}_{\text{cx}}^a = -\nu_{\text{cx}} \left[\frac{1}{2} m (\mathbf{u}_a^2 - \mathbf{u}_i^2) + \frac{3}{2} (\theta_a - \theta_i) \right]. \quad (81)$$

This expression includes the desired first order corrections to our previous second order velocity moment, Eq. (72). Note that the contribution of charge exchange to the flow equations has the same expression as that of elastic intercollisions, Eq. (59). They only differ in their rates, and, again, by the factor of 2.

Finally, we want to suggest an intuitive first order correction to ν_{cx} . The expression for the rate of charge exchange in Eq. (73) depends upon $|\mathbf{u}_a - \mathbf{u}_i|$, and is independent of the thermal velocities of the two components. We suggest that the first order correction should be something like

$$\nu_{\text{cx}} \sim n_i n_a (|\mathbf{u}_i - \mathbf{u}_a| + \mathbf{v}_a^T + \mathbf{v}_i^T) \sigma_{\text{cx}} (\mathbf{v}_a^T + \mathbf{v}_i^T + |\mathbf{u}_i - \mathbf{u}_a|), \quad (82)$$

similar to the case of elastic intercollisions.

VI. DISCUSSION

In summary, we have derived the hydrodynamic equations of motion for a partially ionized plasma, when the charged component, and the neutral component, have different flow velocities and different temperatures. We have started by presenting a general procedure, in the kinetic level, to treat the hydrodynamics of a gas in a general binary mixture, when the interaction between particles of the same species is much stronger than that between particles of different species. This procedure was used to derive the interspecies interaction terms in the flow equations of two mixed gases, of equal atomic masses, with a large difference between their flow velocities. These interaction terms were later refined, by introducing a generalized version of the standard relaxation time approximation. The extended relaxation time approximation was then used to consider various processes of encounters between the two species, and to obtain the moments' equations for each of them. The following processes of interactions were studied: self-collisions, interspecies collisions, ionization, recombination, and charge exchange. The hydrodynamic equations are summarized by the three conservation equations for the mass, Eq. (53), the momentum,

Eq. (55), and the energy, Eq. (57), for ions and atoms, where the Greek indices are replaced by i for ions, and a for atoms. Our results are contained in the source terms of these equations. This is outlined as follows:

(i) The mass equation source is only due to the ionization and recombination processes. The rate of mass transfer from the ions to the atoms is

$$S^{ai} = S_{ir}^{ai}, \quad (83)$$

which is given by Eq. (63). The mass source term for the ions is then $S^{ia} = -S_{ir}^{ai}$.

(ii) The momentum equation source is caused by all the interspecies processes. The rate of momentum transfer from the ions to the atoms is

$$\mathcal{F}^{ai} = \mathcal{F}_{ic}^{ai} + \mathcal{F}_{ir}^{ai} + \mathcal{F}_{cx}^{ai}, \quad (84)$$

with the different drag forces given by Eqs. (61),(64),(80), respectively. An analogous momentum source term is obtained for the ions.

(iii) The energy equation source is also due to all the interspecies processes. The rate of energy transfer from the ions to the atoms is

$$\mathcal{H}^{ai} = \mathcal{H}_{ic}^{ai} + \mathcal{H}_{ir}^{ai} + \mathcal{H}_{cx}^{ai}, \quad (85)$$

where the various energy sources are given by Eq. (58), with α replaced by a and β by i , Eqs. (65) and (81). Again, an analogous energy source term is obtained for the ions.

(iv) Each of the above collision processes is characterized by a single rate. These rates are ν_{sc}^a , ν_{sc}^i , ν_{ic} , ν_{ion} , ν_{rec} , and ν_{cx} , representing the rates of intraspecies encounters, interspecies encounters, ionization reactions, recombination reactions, and charge exchange encounters, respectively. The strength of each coupling process is proportional to its characteristic rate. These rates may be either calculated using a model for the system, or established phenomenologically by comparing to experiment.

We have derived explicit expressions for the rates of elastic collisions and charge exchange reactions. We would now like to relate to some consequences of these expressions.

(i) We have already pointed out that, at the level of the flow equations, the effects of elastic intercollisions and charge exchange reactions differ only in their rates. These rates are given in Eq. (52) for elastic intercollisions and in Eq. (82) for charge exchange. Comparing the two rates we notice that they differ only in the cross section of a single encounter, which is known to be generally bigger for charge exchange. However, the cross section of intercollisions is a constant to good approximation—and is of the order of $\pi(2r_a)^2$ in the hard spheres picture—while that of the charge exchange cross section, i.e., Eq. (67), decreases with increasing impact velocities. In cases of large relative flow, the charge exchange cross section may decrease to values similar to that of elastic intercollisions.

(ii) The expressions for the intercollisions time scales, τ_{ic}^{ai} and τ_{ic}^{ia} , in Eq. (51), together with the expressions for $\tau_{sc}^a(\mathbf{v})$, $\tau_{sc}^i(\mathbf{v})$ (see Sec. IV), allow us to check the consistency of our approximations. Remember that our whole description

is based on the assumption that the effect of self-collisions is much stronger than that of the intercollisions, or as we have expressed it in Sec. II: $\tau_{sc}^a(\mathbf{v}), \tau_{sc}^i(\mathbf{v}) \ll \tau_{ic}^{ai}, \tau_{ic}^{ia}$. Now we have explicit expressions for these time scales: $\tau_{\alpha\alpha}(\mathbf{v})$ stands for the implicit τ_{sc}^a while $\tau_{\alpha\beta}$ or $\tau_{\alpha\beta}(\mathbf{u}_\beta)$ for τ_{ic}^{ai} , etc. When we compare the two relaxation times, as expressed by

$$\frac{\tau_{sc}^{aa}}{\tau_{ic}^{ai}} \sim \frac{n_i \sum_{ai} v_i^T + |\mathbf{u}_a - \mathbf{u}_i|}{n_a \sum_{aa} v_a^T},$$

$$\frac{\tau_{sc}^{ii}}{\tau_{ic}^{ia}} \sim \frac{n_i \sum_{ia} v_i^T + |\mathbf{u}_a - \mathbf{u}_i|}{n_a \sum_{ii} v_i^T}, \quad (86)$$

we observe that to secure our basic assumptions in the case of large relative flow difference, the cross sections ratios \sum_{ai}/\sum_{aa} and \sum_{ia}/\sum_{ii} must be much smaller than the small parameters η_a and η_i .

Finally, we would like to suggest criteria to determine whether the dynamics of a mixture of two gases should be considered as a single fluid, or rather as two separate fluids. We mention two cases in which individual flow equations should be used to each component:

(i) *When an initial value problem is considered for two mixed gases with different flow parameters.* In this case, the flow parameters of the two components, i.e., the flow velocity and kinetic temperature, relax towards common values, on time scales τ_{ic}^{ai} and τ_{ic}^{ia} (or with cx subscripts for charge exchange encounters). The final common flow velocity and kinetic temperature may be determined by considerations of momentum and energy conservation. We conclude that the separate fluids treatment is therefore crucial when considering time scales on the order of (or shorter than) the intercollisions or charge exchange relaxation times.

(ii) *When there is an appreciable difference in the action of external forces on the two gases.* This situation may occur in partially ionized plasmas: when significant electromagnetic forces are present there is a major difference between \mathbf{F}_a and \mathbf{F}_i of Eq. (55). While \mathbf{F}_a is negligible in laboratory plasmas, or is just the force of gravitation in astrophysical plasmas, \mathbf{F}_i also includes the electromagnetic forces [$q(\mathbf{E} + \mathbf{v} \times \mathbf{B}/c)$]. The charge species is then driven differently than the neutral one, and a relative drift develops. We may consider a quasiequilibrium between the different forces acting on each fluid, in which a steady drift is established. A steady relative drift means that the accelerations of the two components, due to their responses to the different external fields, are equal. Using Eq. (55) with Eqs. (61),(64),(80) and Eqs. (52),(82) we may find an equation for the “steady” relative drift, $\Delta \mathbf{u}_{ia}^{\text{steady}}$, in such a quasiequilibrium:

$$\left\{ |\Delta \mathbf{u}_{ia}^{\text{steady}}| m_a \left[n \left[\nu_{cx}(|\Delta \mathbf{u}_{ia}^{\text{steady}}|) + \frac{\nu_{ic}(|\Delta \mathbf{u}_{ia}^{\text{steady}}|)}{2} \right] + n_a \nu_{ion} \right. \right. \\ \left. \left. + n_i \nu_{rec} \right\} = |n_a n_i (\mathbf{F}_i - \mathbf{F}_a) - (n_a \nabla \cdot \mathbf{\Pi}_i - n_i \nabla \cdot \mathbf{\Pi}_a)|, \quad (87)$$

with $n = n_a + n_i$. Equation (87) shows that the rates of the various encounters and reactions, together with the local values of the external forces, pressures, and densities of both components, determine the magnitude of $\Delta \mathbf{u}_{ia}^{\text{steady}}$, and therefore whether the atom and the ions are “stuck” together, or not. Using Eq. (87) one can estimate the expected values of the relative drift in a certain physical situation, and accordingly decide whether to use a single-fluid flow equations or the multiflow equations, which are investigated in the present paper.

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APPENDIX

Consider a mixture of two fluids, which are described by the distribution functions $f_\alpha(\mathbf{r}, \mathbf{v}, t)$, where \mathbf{r} is the position, \mathbf{v} is the velocity, t is the time, and α is the species index. We recall that the hydrodynamic quantities (density, flow velocity, kinetic temperature, etc.) are defined as velocity moments of $f_\alpha(\mathbf{r}, \mathbf{v}, t)$. We define the following moments.

(i) *Number density*,

$$n_\alpha(\mathbf{r}, t) \equiv \int d^3v f_\alpha(\mathbf{r}, \mathbf{v}, t)$$

$$\text{and mass density } \rho_\alpha = m_\alpha \cdot n_\alpha; \quad (\text{A1})$$

(ii) *flow velocity*,

$$\mathbf{u}_\alpha(\mathbf{r}, t) \equiv \left[\int d^3v \mathbf{v} f_\alpha(\mathbf{r}, \mathbf{v}, t) \right] / n_\alpha(\mathbf{r}, t); \quad (\text{A2})$$

(iii) *kinetic energy density*,

$$\begin{aligned} \epsilon_\alpha(\mathbf{r}, t) &\equiv \int d^3v \frac{1}{2} m_\alpha \mathbf{v}^2 f_\alpha(\mathbf{r}, \mathbf{v}, t) \\ &= \frac{1}{2} m_\alpha n_\alpha(\mathbf{r}, t) u_\alpha(\mathbf{r}, t)^2 + \frac{3}{2} n_\alpha(\mathbf{r}, t) \theta_\alpha(\mathbf{r}, t), \end{aligned} \quad (\text{A3})$$

where θ_α is defined as:

(iv) *kinetic temperature*,

$$\theta_\alpha(\mathbf{r}, t) \equiv \left[\frac{1}{3} m_\alpha \int d^3v [\mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)]^2 f_\alpha(\mathbf{r}, \mathbf{v}, t) \right] / n_\alpha(\mathbf{r}, t). \quad (\text{A4})$$

Notice that ϵ_α is the sum of the flow kinetic energy and the “thermal” kinetic energy (the phrase “thermal” is used here in a general sense).

(v) *Stress tensor*,

$$\begin{aligned} [\Pi_\alpha(\mathbf{r}, t)]_{ij} &\equiv m_\alpha \int d^3v [v_i - u_{\alpha,i}(\mathbf{r}, t)] [v_j \\ &\quad - u_{\alpha,j}(\mathbf{r}, t)] f_\alpha(\mathbf{r}, \mathbf{v}, t); \end{aligned} \quad (\text{A5})$$

(vi) *thermal energy flux*,

$$\mathbf{J}_\alpha(\mathbf{r}, t) \equiv \frac{1}{2} m_\alpha^2 \int d^3v [\mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)] [\mathbf{v} - \mathbf{u}_\alpha(\mathbf{r}, t)]^2 f_\alpha(\mathbf{r}, \mathbf{v}, t). \quad (\text{A6})$$

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- [1] R. A. Nemirovsky, A. Ben-Kish, M. Shuker, and A. Ron, *Phys. Rev. Lett.* **82**, 3436 (1999); **83**, 660 (1999).
- [2] D. A. D’Ippolito and J. R. Myra, *Phys. Plasmas* **6**, 519 (1999).
- [3] P. J. Catto, *Phys. Plasmas* **1**, 1936 (1994).
- [4] C. D. Black and E. H. Scott, *Astrophys. J.* **263**, 696 (1982).
- [5] F. H. Shu, *Astrophys. J.* **273**, 202 (1983).
- [6] R. M. Crutcher, T. Ch. Mouschonas, T. H. Troland, and G. E. Ciolek, *Astrophys. J.* **427**, 839 (1994), and references cited therein.
- [7] A. Frank, T. A. Gardiner, G. Delemarter, T. Lery, and R. Betti, *Astrophys. J.* **524**, 947 (1999).
- [8] F. H. Shu, *The Physics of Astrophysics Volume II—Gas Dynamics* (University Science Books, Mill Valley, CA, 1992), Chap. 27.
- [9] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases*, 2nd ed. (Cambridge University Press, Cambridge, England, 1960), Chap. 8.
- [10] J. O. Hirschfelder, C. F. Curtis, and R. B. Bird, *Molecular Theory of Gases and Liquids* (Wiley, New York, 1954).
- [11] H. Grad, in *Handbuch der Physik* (Springer-Verlag, Berlin, 1957), Vol. 12.
- [12] K. Huang, *Statistical Mechanics*, 2nd ed. (Wiley, New York, 1987).
- [13] We assume that the ions and electrons flow together (high plasma frequency, ω_p). We further assume that only the ions exert an effective drag on the atoms. The last assumption is merely the approximation, $m_e \ll m_i, m_a$.
- [14] Note that for a mixture of two gases, the value of β is determined once α is given. However, we use both of them in denoting F , U , Θ , and τ to emphasize that each of these depend on both fluids.
- [15] R. D. Hazeltine, M. D. Calvin, P. M. Valanju, and E. R. Solano, *Nucl. Fusion* **32**, 3 (1992).
- [16] D. Rapp and W. E. Francis, *J. Chem. Phys.* **37**, 2631 (1962).
- [17] R. Hegerberg, M. T. Elford, and H. R. Skullerud, *J. Phys. B* **15**, 797 (1982).
- [18] A symmetric resonant charge exchange reaction is a charge transfer between an atom A and its first ion A^+ , expressed by $A + A^+ \rightarrow A^+ + A$.